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Acoustic wave propagation and stochastic effects in metamaterial absorbers

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We show how stochastic variations of the effective parameters of anisotropic structured metamaterials can lead to increased absorption of sound. For this, we derive an analytical model based on the Bourret approximation and illustrate the immediate connection between material disorder and attenuation of the averaged field. We demonstrate numerically that broadband absorption persists at oblique irradiation and that the influence of red noise comprising short spatial correlation lengths increases the absorption beyond what can be archived with a structured but ordered system. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4892011]

The acoustical properties of structured materials have recently been the focus of intense attention due to the implications to new phononic devices. Beyond metamaterial related disciplines such as negative refraction or subwavelength imaging for acoustic waves, near complete absorption of sound by tailored materials are exciting tremendous interests due to numerous theoretical predictions and experimental realizations. Among these efforts, we can name the realization of an ultra-thin membrane structure supporting near-complete acoustic absorption for selective frequencies.[1] Broadband absorption on the contrary, demands geometries approaching the size of the wavelengths of interest and has been realized both by lattices of microperforated cylindrical shells and porous lamella-crystals.[2,3] These systems that are providing attenuation of sound of extended bandwidth and for various angles of irradiation are based on the composition or ordered crystal units. Disordered structures, on the other hand, such as randomly layered media or phononic lattices containing disorder, have in many years faced intense interest due to the richness of the physical properties.[4]

A great amount of literature focuses on the physics of localization phenomena in strongly scattered media. The determination of quantities such as the frequency dependent localization length, diffusion coefficients, and the correlation of localized states with the spectral position of band gaps are just a few among many questions, which have been addressed to explore new effects not only in the framework of acoustics but also for the case of light and Schrödinger waves.[5–7] In the same context, transient pulses are widely used in geophysical[8,9] and industrial applications to probe and characterize random materials. These investigations have demonstrated that strong scattering materials can cause substantial resonances and influence the flow of the wave energy. In fact, it is known that strong scattering caused by a significant material disorder results in large attenuation of the wave, even within a lossless material scenario.[10,11] Since sound waves are exponentially attenuated in such media with a given localization length determined via the degree of disorder, it follows naturally that sound can be trapped more efficiently within a lossy material if additional disorder is introduced.

In this work, we investigate an effective acoustic absorber consisting of a lossy anisotropic fluid slab backed by a rigid support[3] and introduce stochastic variations in the spatial material dependence to examine the overall impact on broadband absorption. Within this framework, we construct an analytical model based on the Bourret approximation comprising small correlation distances ξ_c. Using this method, we demonstrate how the wave equation for anisotropic fluids can be recast into systems comprising intrinsic dissipation measured by the stochastic process that is derived via the Bourret integral equation for slowly varying solutions of short correlation distances (times).[12] In an earlier work, the anisotropic fluid slab has been fabricated and constructed out of a porous lamella-crystal and exhibit extraordinary broadband absorption for any direction of incidence sound.[3] Introducing stochastic colored noise into these systems improves the overall absorption and suggests utile and improved tuning strategies for efficient acoustic concealing in metamaterials.

We begin the analysis by plotting the effective parameters ρ_x, ρ_z, and K, which are the mass, densities, and the bulk modulus, respectively, such as the absorption within the long wavelength regime based on an effective medium theory (EMT). The anisotropic structure is a crystal mounted on a rigid support as illustrated in Fig. 1(a). The EMT consists in averaging the effective mass density with tensor components $\rho^x_{\text{eff}} = \text{f}\rho_l + (1-\text{f})\rho_0$ and $\frac{1}{\rho^z_{\text{eff}}} = \frac{\text{f}}{\rho_l} + \frac{1-\text{f}}{\rho_0}$ as well as the scalar effective bulk modulus $\frac{1}{K_{\text{eff}}} = \frac{\text{f}}{K_l} + \frac{1-\text{f}}{K_0}$ within the unit cell as depicted in Fig. 1(a). Subscripts *l* indicate material parameters in the lamella and *0* in the fluid background. These effective parameters are employed to calculate the overall absorption containing, e.g., $r_{21} = (\rho^z_{\text{eff}}k_z - \rho_0 q_z)/(\rho^z_{\text{eff}}k_z + \rho_0 q_z)$ the reflection coefficient at the first interface. Here, k_z and q_z denote the wavenumbers perpendicular to the layer interface in free space and within the metamaterial, respectively, more details are presented elsewhere.[13] As we have reported in earlier works based on experimental realizations, we apply dissipative material properties of compacted and compressed polyurethane for the lamella material.[3] These lamellas have the width W and length L and are arranged into a crystal with lattice constant a. We use a rigorous wave expansion approach in order to derive exact complex

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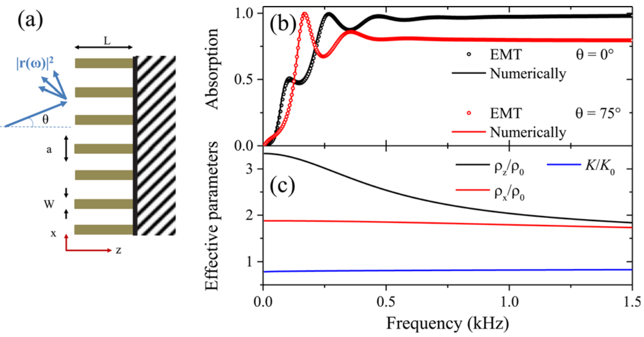

FIG. 1. (a) Schematics of the structure under investigation. A lossy material is cut into lamella and entirely surrounded by air (ρ_0, K_0), where free-space wave irradiation is incident from left to right. We denote the size of the unit cell a, the width of the lamella W, and the filling fraction f = W/a. (b) We lock the parameters f = 70% and $L = 0.5$ m and plot the spectral absorption both at normal and oblique incidence ($\theta = 75°$) obtained by numerical simulations and an EMT.[13] (c) For the same configurations as above we calculate the effective anisotropic parameters.

scattering parameters and conduct a quantitative comparison with the EMT. Spectrally we compute the absorption both for normal and oblique ($\theta = 75°$) incidence and obtain quite good agreement within a regime, where contributions of diffraction can be safely omitted as seen Fig. 1(b). It is thus verified that the porous lamella-crystal within the given frequency range of interest, safely can be treated as a metamaterial. In this context, we plot the effective constitutive parameters in Fig. 1(c), demonstrating uniaxial anisotropy for the effective mass densities that are converging toward isotropy for higher frequencies and a spectrally slowly varying bulk modulus K.

From this point on we can safely consider the complex wave interaction of the lossy crystal to be adequately represented by a homogeneous effective slab. A good starting point is given by assuming that the effective bulk modulus is not homogeneous but instead a stochastic function of space along x and we seek a separable solution to the following anisotropic form of the wave equation

$$\frac{1}{\rho_x}\frac{\partial^2 p}{\partial x^2} + \frac{1}{\rho_z}\frac{\partial^2 p}{\partial z^2} - \frac{1}{K(x)}\frac{\partial^2 p}{\partial t^2} = 0. \tag{1}$$

The construction of the bulk modulus $K(x) = \langle K\rangle + \tilde{K}$ consists of an averaged and a Gaussian noise contribution. Since we have restricted the present analysis to stochastic material variations along x, it is clear that oblique irradiation will be regarded here. Before arriving there, we assume the solution of Eq. (1) to be of a separable form $p(x,z) = p_x(x)p_z(z)$ $\exp(i\omega t)$ and after substituting it above, we obtain

$$\frac{1}{\rho_x}\frac{d^2 p_x}{dx^2} + \left(\frac{\omega^2}{K(x)} - \delta\right)p_x = 0$$
$$\frac{1}{\rho_z}\frac{d^2 p_z}{dz^2} + \delta p_z = 0, \tag{2}$$

with $\delta = q_z^2/\rho_z$ a separation constant such that we obtain a general dispersion relationship containing stochastic variations

$$\frac{k_x^2}{\rho_x} = \frac{\omega^2}{K(x)} - \delta. \tag{3}$$

The z component of the solution can be expressed as a wave similar to the one of a fixed-free vibrating string because of the rigid back support at $z = L$ and the imposed continuity in pressure and velocity at the free air interface at $z = 0$, see Fig. 1(a). The x component of the solution however must be treated with care to which we must also consider the amount of disorder. In the following, we shall derive an approximate solution that assumes very short correlation distance for the fluctuation. Physically, this means that variations in $K(x)$ are to be expected on a microscopic level originating from small material variations in space. An extension considering fluctuations of the effective mass densities and a long correlation distance is a topic for further investigation. The x dependent equation of Eq. (2) is a linear stochastic differential equation that can be recast into the canonical form consisting of two ordinary differential equations after transforming the spatial dependence with $\tilde{x} = \mu x$. This transformation consists in defining an averaged wavevector along x of the following form:

$$\mu = \sqrt{\rho_x}\sqrt{\frac{\omega^2}{\langle K\rangle} - \delta}, \tag{4}$$

hence, without loss of generality the x dependent equation of Eq. (2) can be expressed in a more familiar non-dimensionalized form

$$\frac{d\bar{p}_x}{d\tilde{x}} = (\mathbf{A}_0 + \epsilon\eta(x)\mathbf{A}_1)\bar{p}_x, \tag{5}$$

where $\bar{p}_x = (p_x^1, p_x^2)$, $p_x^2 = \frac{dp_x^1}{dx}$, ϵ is the amplitude of the normalized stochastic noise function $\eta(x)$ and $\mathbf{A}_0$ and $\mathbf{A}_1$ are matrices representing the basic state and the random process, respectively. By considering only lowest-order terms in the perturbation expansion for short correlation distances, we apply Bourrets integral equation in solving for the averaged pressure $\langle\bar{p}_x\rangle$.[12] After some algebra, it is convenient to present the integral equation on a differential form in the averaged pressure, hence the Bourret approximation reads

$$\frac{d\langle\bar{p}_x\rangle}{d\tilde{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\langle\bar{p}_x\rangle + \frac{\epsilon^2\alpha}{2}\begin{bmatrix} 0 & 0 \\ c_1 & -c_2 \end{bmatrix}\langle\bar{p}_x\rangle, \tag{6}$$

where

$$c_1 = \frac{2\xi_c^2}{1+4\xi_c^2} \quad \text{and} \quad c_2 = \frac{4\xi_c^3}{1+4\xi_c^2} \tag{7}$$

are constants and $\eta(x)$ contains statistics in terms of Gaussian red noise

$$\langle\eta(x)\rangle = 0 \quad \text{and} \quad \langle\eta(x)\eta(x-\xi)\rangle = \alpha e^{\frac{-\omega|\xi|}{\langle c\rangle\xi_c}}, \tag{8}$$

where α is the variance and $\langle c\rangle = \sqrt{\langle K\rangle/\rho_x}$. Within the limits where $\epsilon\xi_c \ll 1$, we can rewrite Eq. (6) into a damped differential equation

$$\frac{d^2\langle p_x\rangle}{d\tilde{x}^2} + 2\gamma\frac{d\langle p_x\rangle}{d\tilde{x}} + (1 - \gamma/\xi_c)\langle p_x\rangle = 0, \tag{9}$$

where the damping term $\gamma = (\epsilon^2 \alpha \xi_c^3)/(1 + 4\xi_c^2)$. The fluctuations in $K(x)$ consequently cause a damping in the averaged field $\langle p_x \rangle$ along x. We have restricted ourselves to consider only the case of colored noise of short correlation lengths. Immediately we see, that when operating within the white noise limit, $\xi_c \to 0$, γ vanishes and waves propagate stochastically undamped along x. Interestingly, we could also have treated the same problem by considering inhomogeneous materials along x, where the bulk modulus $K(x)$ is stochastically varied in space. We begin by writing the governing inhomogeneous wave equation

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho(x)}\frac{\partial p}{\partial x}\right) + \frac{1}{\rho(x)}\frac{\partial^2 p}{\partial z^2} + \frac{\omega^2}{K(x)}\frac{\partial^2 p}{\partial t^2} = 0. \quad (10)$$

The Bourret approximation is then made, i.e., stochastic variations occur on a small length scale compared to the period a, and a stochastically averaged equation is obtained:

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho(x)}\frac{\partial\langle p\rangle}{\partial x}\right) + \frac{2\epsilon^2\alpha\xi_c^3\omega}{\rho(x)\langle c\rangle}\frac{\partial\langle p\rangle}{\partial x} + \frac{\omega^2}{\langle K\rangle}\left(1 - \frac{\lambda^2\langle K\rangle}{\rho(x)\omega^2} - \epsilon^2\alpha\xi_c^2\right)\langle p\rangle = 0, \quad (11)$$

where λ^2 is a separation constant due to the homogeneity along the z direction and $\langle c\rangle = \sqrt{\langle K\rangle/\rho(x)}$ and $\epsilon\eta(x) = \tilde{K}/\langle K\rangle$. Note that Eq. (9) after denormalization agrees with Eq. (11) in the metamaterial approximation. This finding is important since it proves that the averaged field is damped via stochastic noise and that the damping level is unaffected by first homogenizing in a metamaterial sense and then stochastically averaging or vice versa. We recall that, in the former case, the analysis was initiated by using effective parameters to demonstrate damping of the averaged field. In the latter case, however, it can be seen that the inhomogeneous wave equation, Eq. (11), reveals equivalent damping by the same stochastic mechanism. This is consistent since noise is imposed onto an averaged bulk modulus in both procedures.

Henceforth, we proceed by employing Eq. (9) and solving the complex scattering problem. In order to express the overall absorption of the anisotropic structure comprising stochastic fluctuations of $K(x)$, we first write the combined solution of $p(x,z) = \langle p_x(x)\rangle p_z(z)\exp(i\omega t)$ as

$$p(x,z) = 2B\cos\left(q_z(z-L)\right)e^{-\gamma\mu x}e^{i(1-\gamma/2\xi_c)\mu x}e^{i\omega t}, \quad (12)$$

where B is the complex wave amplitude. If we define $\beta = \frac{q_z\rho_0}{k_z\rho_z}$, which is a quantity much similar to the structure impedance mismatch compared to the background fluid and $k_z = k_0\sqrt{1-\sin^2\theta}$, we proceed by matching free-space sound irradiation onto the structured stochastic metamaterial. Following this, it is straightforward to derive the analytical reflection coefficient

$$r(\omega) = \frac{2\cos(q_z L)}{\cos(q_z L) - i\beta\sin(q_z L)} - 1, \quad (13)$$

from which we can deduce the absorption $A = 1 - |r(\omega)|^2$.

In Fig. 2, we plot the spectral absorption which has been derived from Eq. (13) and the conservation of energy. As mentioned earlier, since we are considering only stochastic variations along x for the bulk modulus $K(x)$ we must irradiate the anisotropic slab from relative sharp angles to see a noticeable effect of the material fluctuations. This will not cause any troubles since we have already tested the validity of the EMT to work for some specific directions as seen in Fig. 1(b). Not shown here, is the case where sound is coming in at normal incidence and the influence of noise is zero since the wave does not have a vector component parallel to the interface, the direction along which noise has been introduced. Even though the damping term in Eq. (9) is small; at oblique incidence, we predict that the contribution of noise indeed alters the spectral response of the absorption as plotted in Fig. 2 for various values of γ. In this plot, we also show the case comprising white noise, $\xi_c \to 0$, that does not change the absorption since the mean field $\langle p_x\rangle$ remains unaffected within this scenario. Although we are restricted to short correlation lengths, we simulate for various values of γ and find that the absorption increases only slightly as shown in Fig. 2. This behaviour is expected since by the introduction of small material fluctuations we add an additional loss channel into the already lossy material as can be clearly seen in the inset of Fig. 2, where $\langle p_x\rangle$ as a function of distance is attenuated for various values of γ.

We now know that stochastic material fluctuations to some extent influence the spectral absorption for acoustically thick structures. Next, we simulate the influence of noise on the absorption when we vary the length L of the metamaterial slab to find out whether a thinner slab performs equally well. In Fig. 3(a), we compute the white noise regime and predict some oscillations that scale inversely with length. If we add some red noise to the structure with the parameter, e.g., $\gamma = 0.15$, it can be seen in Fig. 3(b) that absorption is increased for a relatively broad range of frequencies also where L is relatively small. However, contrary to this, we predict that the oscillations overall appear more pronounced

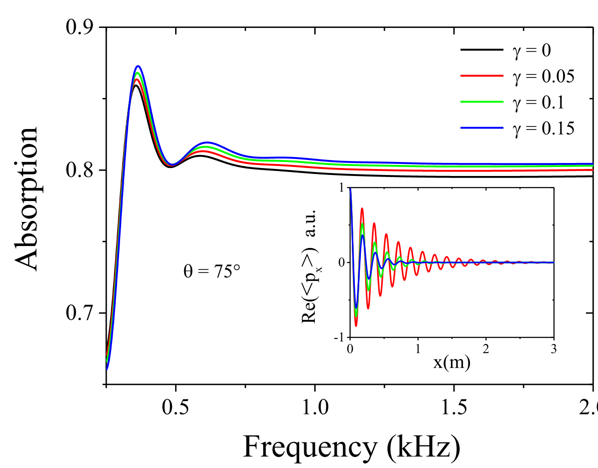

FIG. 2. Absorption spectra for sound irradiating the anisotropic stochastic metamaterial at oblique incidence, $\theta = 75°$. We have considered geometrical parameters as in the previous case where f = 70% and $L = 0.5$ m. The influence of changing the variance is simulated and expressed via the damping term γ. The correlation length $\xi_c = 1.0$ and $\epsilon = 0.1$. Inset: $\langle p_x\rangle$ as a function of x.

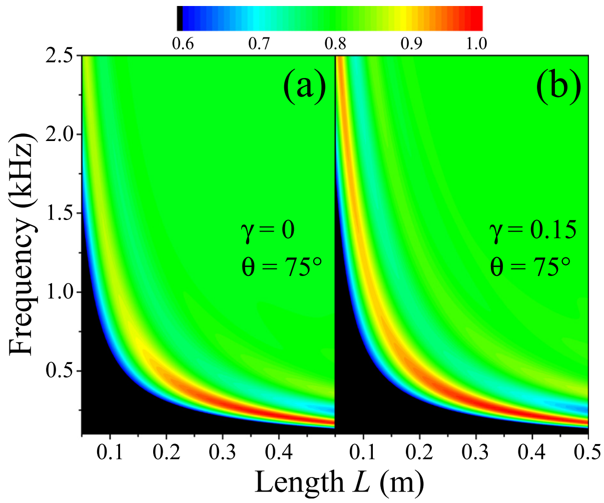

FIG. 3. With the same geometrical parameters as in the latter simulations, we plot the absorption versus frequency and slab length L. We study the influence of the length both in the limit of (a) white ($\xi_c \rightarrow 0$) and (b) red noise where $\epsilon\xi_c \ll 1$.

such that regions in the plotted contour emerge with both increased but also lowered absorption of sound.

In conclusion, within the limits of short disorder correlation distances, that is, a small amount of disorder, we developed a theoretical model based on the Bourret approximation to understand the influence of stochastic noise in some material parameters. It has been shown analytically that the averaged field along the axis of stochastic variations is attenuated, and we verify this by numerical simulations predicting that microscopical noise in lossy metamaterials increase the overall absorption. The influence of stochastic noise on the Wigner-Smith delay time[3] or similarly the effective impedance in the long wavelength regime is of relevance for future studies in order to understand the formation of pronounced oscillations of the spectral absorption in the presence of red noise. Also it is of high interest to overcome the present restrictions of short correlation distances by considering Monte Carlo simulations.

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